Journal of Global Optimization (2006) 34: 597–607
DOI 10.1007/s10898-005-8565-z

Trees with Equal Domination and Restrained Domination numbers

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(Received 21 June 2004; accepted in revised form 4 June 2005)

Abstract. Let $G=(V,E)$ be a graph and let $S\subseteq V$. The set S is a packing in G if the vertices of S are pairwise at distance at least three apart in G. The set S is a dominating set (DS) if every vertex in $V-S$ is adjacent to a vertex in S. Further, if every vertex in $V-S$ is also adjacent to a vertex in $V-S$, then S is a restrained dominating set (RDS). The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a DS of G, while the restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of G. The graph G is γ-excellent if every vertex of G belongs to some minimum DS of G. A constructive characterization of trees with equal domination and restrained domination numbers is presented. As a consequence of this characterization we show that the following statements are equivalent: (i) T is a tree with $\gamma(T)=\gamma_r(T)$; (ii) T is a γ-excellent tree and $T\neq K_2$; and (iii) T is a tree that has a unique maximum packing and this set is a dominating set of T. We show that if T is a tree of order n with ℓ leaves, then $\gamma_r(T)\leqslant(n+\ell+1)/2$, and we characterize those trees achieving equality.

Mathematics Subject Classification: 05C69

Key words: domination, domination excellent trees, restrained domination

1. Introduction

In this paper, we continue the study of restrained domination in trees started in [3,5,7]. For a graph $G=(V,E)$, a set S is a *dominating set* if every vertex in $V-S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We call a dominating set of cardinality $\gamma(G)$ a $\gamma(G)$-*set* and use similar notation for other parameters. An *independent dominating set* is a dominating set that is independent, and the *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set of G. Domination and its many variations have been surveyed in [9,10].

In this paper we study a variation on the domination theme called restrained domination, introduced by Telle and Proskurowski [14], albeit indirectly, as vertex partitioning problem and further studied in [3–5,7,8]. A set $S \subseteq V$ is a *restrained dominating set* (RDS) if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. Every graph has a RDS, since $S = V$ is such a set. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of G. Clearly, $\gamma(G) \leqslant \gamma_r(G)$. If $\gamma(G) = \gamma_r(G)$, then we call G a (γ, γ_r)-graph.

A graph G is called γ-excellent (respectively, i-excellent) if every vertex of G belongs to some $\gamma(G)$-set (respectively, some $i(G)$-set). Results on γ-excellent trees and i-excellent trees can be found in [1,6,11,13] and elsewhere.

In general we follow the notation and graph theory terminology in [2,9]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. For any vertex $v \in V$, the *open neighborhood of* v is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A vertex $w \in V$ is a *private neighbor of* v (with respect to S) if $N[w] \cap S = \{v\}$; and the *private neighbor set of* v *with respect to* S, denoted $\mathrm{pn}(v, S)$, is the set of all private neighbors of v. If S is a $\gamma(G)$-set, then $\mathrm{pn}(v, S) \neq \emptyset$ for each $v \in S$. If $A, B \subseteq V$, then the set B is said to *dominate* the set A if $A \subseteq N[B]$. In particular, if $A = V$, then B is a dominating set of G.

For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is adjacent to at least two leaves. A *double star* is a tree with exactly two vertices that are not leaves. A tree on one vertex is denoted by K_1 and a tree on two vertices by K_2.

We will need the following fact from [12]. A subset $S \subseteq V$ is a *packing* in G if the vertices of S are pairwise at distance at least three apart in G. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G.

THEOREM 1. (Moon and Meir [12]) *For a tree* T, $\gamma(T) = \rho(T)$.

Our aim in this paper is twofold: First to establish a sharp upper bound on the restrained domination number of a tree in terms of its order and the number of leaves, and second to give a characterization of (γ, γ_r)-trees. More precisely, we show that if T is a tree of order n with ℓ leaves, then $\gamma_r(T) \leqslant (n + \ell + 1)/2$, and we characterize those trees achieving equality. A constructive characterization of (γ, γ_r)-trees is presented. As a consequence of this characterization we show that the following statements are equivalent: (i) T is a (γ, γ_r)-tree; (ii) T is a γ-excellent tree and $T \neq K_2$; and

(iii) T is a tree that has a unique $\rho(T)$-set and this set is a dominating set of T.

2. Upper bounds

Since every leaf of a tree belongs to every RDS in the tree, a natural question is to find a sharp upper bound on the restrained domination number of a tree in terms of its order and the number of leaves. The following result establishes such a bound.

THEOREM 2. *If T is a tree of order n with ℓ leaves, then $\gamma_r(T) \leqslant (n+\ell+1)/2$ with equality if and only if T is a nontrivial star.*

Proof. We proceed by induction on n. The base case when $n=1$ is trivial. Assume then that $n \geqslant 2$ and that the result holds for all trees of order less than n. Let T be a tree of order n with ℓ leaves. If T is a star, then $n=\ell+1$ and $\gamma_r(T)=(n+\ell+1)/2$. If T is a double star, then $n=\ell+2$ and $\gamma_r(T)=l<(n+\ell+1)/2$. Hence we may assume that $\text{diam}(T) \geqslant 4$ (and so, $n \geqslant 5$).

Suppose T has a strong support vertex w. Let v be a leaf-neighbor of w, and let $T'=T-v$ have order n' with ℓ' leaves. Then, $n'=n-1$ and $\ell'=\ell-1$. Since T is not a star and since w is a support vertex of degree at least 2 in T', the tree T' is not a nontrivial star. Applying the inductive hypothesis to T', $\gamma_r(T') \leqslant (n'+\ell')/2 \leqslant (n+\ell-2)/2$. Any $\gamma_r(T')$-set can be extended to a RDS of T by adding to it the vertex v, whence $\gamma_r(T) \leqslant (n+\ell)/2$, as desired. Thus we may assume that T has no strong support vertex.

Let T be rooted at a leaf r of a longest path P. Let P be a r–u path, and let v be the neighbor of u. Further, let w denote the parent of v on this path, and let y denote the parent of w. Then, u is a leaf of T and $\deg_T(v)=2$. We consider two possibilities depending on the degree of w.

Case 1. Suppose $\deg_T(w)=2$. Let $T'=T-\{u,v,w\}$ have order n' and ℓ' leaves. Then, $n'=n-3 \geqslant 2$. Suppose y is a leaf of T'. Then, $\ell'=\ell$. Applying the inductive hypothesis to T', $\gamma_r(T') \leqslant (n'+\ell'+1)/2=(n+\ell-2)/2$. Any $\gamma_r(T')$-set can be extended to a RDS of T by adding to it the vertex u, whence $\gamma_r(T) \leqslant (n+\ell)/2$, as desired. Hence we may assume that y is not a leaf in T'. Thus, $\ell'=\ell-1$ and since y cannot be a strong support vertex, T' is not a star. Applying the inductive hypothesis to T', $\gamma_r(T') \leqslant (n'+\ell')/2 \leqslant (n+\ell-4)/2$. Let S' be a $\gamma_r(T')$-set. If $y \in S'$, let $S=S' \cup \{u\}$. If $y \notin S'$, let $S=S' \cup \{u,v\}$. In both cases, S is a RDS of T, whence $\gamma_r(T) \leqslant \gamma_r(T')+2 \leqslant (n+\ell)/2$.

Case 2. Suppose $\deg_T(w) \geqslant 3$. If w is a support vertex, let z denote the leaf-neighbor of w, and let $T' = T - z$ have order n' with ℓ' leaves. Then, $n' = n - 1$ and $\ell' = \ell - 1$. Since $\text{diam}(T') \geqslant 4$, T' is not a star. Hence applying the inductive hypothesis to T', $\gamma_r(T') \leqslant (n' + \ell')/2 = (n + \ell - 2)/2$. Any $\gamma_r(T')$-set can be extended to a RDS of T by adding to it the vertex z, whence $\gamma_r(T) \leqslant (n + \ell)/2$, as desired. Thus we may assume that every child of w is a support vertex of degree 2.

Let $k = \deg_T(w) - 1 \geqslant 2$. Let $T^* = T - V(T_w)$ have order n^* and ℓ^* leaves. Then, $n^* = n - 2k - 1$. Since $\text{diam}(T) \geqslant 4$, T^* is a nontrivial tree. If T^* is a star, then our earlier assumptions imply that $T^* \in \{P_2, P_3\}$ and that y is a leaf of T^*, and the desired result follows readily. Hence we may assume that T^* is not a star. Applying the inductive hypothesis to T^*, $\gamma_r(T^*) \leqslant (n^* + \ell^*)/2$.

Suppose y is a leaf of T^*. Then, $\ell^* = \ell - k + 1$, and so $\gamma_r(T^*) \leqslant (n + \ell - 3k)/2$. Any $\gamma_r(T^*)$-set can now be extended to a RDS of T by adding to it the k leaves in the subtree T_w, whence $\gamma_r(T) \leqslant (n + \ell - k)/2 < (n + \ell)/2$, as desired. On the other hand, if y is not a leaf of T^*, then $\ell^* = \ell - k$, and so $\gamma_r(T^*) \leqslant (n + \ell - 3k - 1)/2$. Any $\gamma_r(T^*)$-set can now be extended to a RDS of T by adding to it the k leaves in the subtree T_w and the vertex v, whence $\gamma_r(T) \leqslant (n + \ell + 1 - k)/2 < (n + \ell)/2$, as desired. □

3. (γ, γ_r)-trees

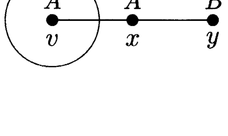
Several characterizations of (γ, γ_r)-trees are given in [7]. The characterization we present here is a constructive characterization using labellings that is simpler than those presented in [7].

We define a *labeling* of a tree T as a function $S: V(T) \to \{A, B\}$. The label of a vertex v is also called its *status*, denoted $\text{sta}(v)$. A labelled tree is denoted by a pair (T, S). We denote the sets of vertices of status A and B by $S_A(T)$ and $S_B(T)$, respectively, or simply by S_A and S_B if the tree T is clear from context.

By a *labeled* K_1 we shall mean a K_1 whose vertex is labelled with status B. Let $\mathcal{T}$ be the family of trees that can be labelled so that the resulting family of labelled trees contain a labeled K_1 and is closed under the two operations $\mathcal{O}_1$ and $\mathcal{O}_2$ listed below, which extend the tree T by attaching a tree to the vertex $v \in V(T)$, called the *attacher*.

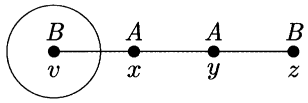
- **Operation $\mathcal{O}_1$.** Attach to a vertex v of status A a path v, x, y where $\text{sta}(x) = A$ and $\text{sta}(y) = B$.

$\mathcal{O}_1$:



- **Operation $\mathcal{O}_2$.** Attach to a vertex v of status B a path v, x, y, z where $\text{sta}(x) = \text{sta}(y) = A$ and $\text{sta}(z) = B$.

$\mathcal{O}_2$:



Before presenting our main result of this section, we prove the following three lemmas.

LEMMA 3. *Let $T \in \mathcal{T}$. Then the following five properties hold:*

(a) *the set S_B is a packing;*
(b) *every $v \in S_A$ is adjacent to at least one vertex in S_A and to exactly one vertex in S_B;*
(c) *S_B is a $\gamma(T)$-set, a $\rho(T)$-set, and a $\gamma_r(T)$-set;*
(d) *S_B is the unique $\gamma_r(T)$-set;*
(e) *S_B is the unique $\rho(T)$-set.*

Proof. Properties (a) and (b) are immediate from the way in which the family $\mathcal{T}$ is constructed. These two properties imply that S_B is a RDS of T. Hence, by Theorem 1, $|S_B| \leqslant \rho(T) = \gamma(T) \leqslant \gamma_r(T) \leqslant |S_B|$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_r(T) = |S_B|$ and property (c) follows.

To prove property (d), let $T = (V, E)$ and let R be a $\gamma_r(T)$-set. Since R is a dominating set, $|R \cap N[v]| \geqslant 1$ for each $v \in S_B$. By (c), $|R| = |S_B|$ and the sets $R \cap N[v]$, where $v \in S_B$, partition $V(T)$. Consequently, $|R \cap N[v]| = 1$ for each $v \in S_B$. We show that $R = S_B$. Suppose that R contains a vertex $v_1 \in S_A$. Let v_2 be the unique vertex in S_B adjacent to v_1. Then, $R \cap N[v_2] = \{v_1\}$. Since R is a RDS, there is a vertex $v_3 \in V - R$ adjacent to v_2. Since the set S_B is a packing, $v_3 \in S_A$. Now since R is a dominating set, there is a vertex $v_4 \in R$ adjacent to v_3. Necessarily, $v_4 \in S_A$. Let v_5 be the unique vertex in S_B adjacent to v_4. Then, $R \cap N[v_5] = \{v_4\}$. Since R is a RDS, there is a vertex $v_6 \in S_A - R$ adjacent to v_5. Continuing in this way, we construct an infinite path $v_1, v_2, v_3, \ldots$, contradicting the fact that T has finite order. Hence, $R = S_B$.

To prove property (e), we proceed by induction on $|S_B(T)|$. The base case when $|S_B| = 1$ is immediate since then T is a labelled K_1. Let $k \geqslant 2$ and suppose that for all trees $T' \in \mathcal{T}$ with $|S_B(T')| < k$ that $S_B(T')$ is the unique $\rho(T')$-set. Let $T \in \mathcal{T}$ have $|S_B| = k$. Then, T can be obtained from a sequence $T_1, T_1, \ldots, T_m = T$ of trees, where T_1 is a labelled K_1 and $T = T_m$, and T_{i+1} can be obtained from T_i by operation $\mathcal{O}_1$ or $\mathcal{O}_2$ for $i = 1, \ldots, m-1$. Let $T' = T_{m-1}$, and let D be a $\rho(T)$-set. Then, $T' \in \mathcal{T}$. We consider two possibilities depending on whether T is obtained from T' by operation $\mathcal{O}_1$ or $\mathcal{O}_2$.

Case 1. T is obtained from T′ by operation $\mathcal{O}_1$. Suppose T is obtained from T' by adding a path x, y and the edge vx where $v \in V(T')$ and $\text{sta}(v) = A$. Hence, $\text{sta}(x) = A$ and $\text{sta}(y) = B$. By property (c), $\rho(T) = |S_B(T)|$. Since D is a maximum packing, $|D| = |S_B|$ and $|D \cap \{v, x, y\}| = 1$. If $v \in D$, then D is a packing in T' of cardinality $|S_B(T)| = |S_B(T')| + 1$, and so $\rho(T') \geqslant |S_B(T')| + 1$, contradicting property (c). Hence, $|D \cap \{x, y\}| = 1$. Let $D' = D \cap V(T')$. Then, $|D'| = |S_B(T)| - 1 = |S_B(T')|$. By property (c), $\rho(T') = |S_B(T')|$, and so D' is a $\rho(T')$-set. Applying the inductive hypothesis to T', we have $D' = S_B(T')$. By property (b), the vertex v is adjacent to a vertex in $S_B(T')$, and so $x \notin D$. Thus, $y \in D$, whence $D = S_B(T') \cup \{y\} = S_B$, as desired.

Case 2. T is obtained from T′ by operation $\mathcal{O}_2$. Suppose T is obtained from T' by adding a path x, y, z and the edge vx, where $v \in V(T')$ and $\text{sta}(v) = B$. Hence, $\text{sta}(x) = \text{sta}(y) = A$ and $\text{sta}(z) = B$. Since D is a maximum packing, $|D \cap \{x, y, z\}| = 1$. Let $D' = D \cap V(T')$. Then, $|D'| = |S_B(T)| - 1 = |S_B(T')|$, and so by property (c), D' is a $\rho(T')$-set. Applying the inductive hypothesis to T', we have $D' = S_B(T')$. In particular, $v \in D'$, whence $D = S_B(T') \cup \{z\} = S_B$, as desired.

In both Cases 1 and 2, we have $D = S_B$, as desired. This establishes property (e). □

LEMMA 4. *If a tree T has a unique $\rho(T)$-set and this set is a dominating set of T, then T is a (γ, γ_r)-tree.*

Proof. Let $T = (V, E)$ and let S be the unique $\rho(T)$-set that is also a dominating set of T. Let $u \in V - S$. Since S is a dominating set, u is dominated by a vertex $v \in S$. By the uniqueness of S, the set $(S - \{v\}) \cup \{u\}$ is not a packing in T. Thus the vertex u must be at distance 2 from some vertex of $S - \{v\}$, and therefore u is adjacent to some other vertex of $V - S$. Hence every vertex in $V - S$ is adjacent to some other vertex of $V - S$, whence the dominating set S of T is also a RDS of T. Thus, $\gamma_r(T) \leqslant |S| = \rho(T) = \gamma(T) \leqslant \gamma_r(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_r(T)$, i.e., T is a (γ, γ_r)-tree. □

LEMMA 5. *If T is a (γ, γ_r)-tree, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order n of a (γ, γ_r)-tree T. The result holds true for $T = K_1$. This establishes the base case. Assume then that $n \geqslant 2$ and that if T' is a (γ, γ_r)-tree of order less than n, then $T \in \mathcal{T}'$. Let T be a (γ, γ_r)-tree of order n. Then, T has no strong support vertex and every $\gamma_r(T)$-set contains all the leaves of T and no support vertex of T. Further, by Theorem 2 in [7], every $\gamma_r(T)$-set is a packing.

Since no star is a (γ, γ_r)-tree, $\text{diam}(T) \geqslant 3$. If $\text{diam}(T) = 3$, then $T = P_4$ and the desired result holds. Hence we may assume that $\text{diam}(T) \geqslant 4$. Let T be rooted at an leaf r of a longest path P. Let P be a r–u path, and let v be the neighbor of u. Further, let w denote the parent of v on this path, and let y denote the parent of w. Then, u is a leaf of T and $\deg_T(v) = 2$. Let S be a $\gamma_r(T)$-set. Then, S is a packing in T containing all the leaves. In particular, $u \in S$ and $\{v, w\} \cap S = \emptyset$. We consider two possibilities depending on the degree of w.

Case 1. Suppose $\deg_T(w) = 2$. Then, $y \in S$. Let $T' = T - \{u, v, w\}$. Then, $\gamma(T') = \gamma(T) - 1$. Since $S - \{u\}$ is a RDS of T', $\gamma_r(T') \leqslant |S| - 1 = \gamma_r(T) - 1$. Hence, $\gamma_r(T') \geqslant \gamma(T') = \gamma(T) - 1 = \gamma_r(T) - 1 \geqslant \gamma_r(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T') = \gamma_r(T')$ and $S - \{u\}$ is a $\gamma_r(T')$-set. Thus, T' is a (γ, γ_r)-tree. By the induction hypothesis, $T' \in \mathcal{T}$. By Lemma 3, $S_B(T')$ is the unique $\gamma_r(T')$-set. Thus, $S - \{u\} = S_B(T')$. Since $y \in S$, the vertex y has status B in T'. Hence by operation $\mathcal{O}_2$, our labelling of T' can be extended to a labelling of T so that $T \in \mathcal{T}$.

Case 2. Suppose $\deg_T(w) \geqslant 3$. Let $T' = T - \{u, v\}$. Since w is itself a support vertex or is adjacent to a support vertex other than v, it follows readily that $\gamma(T') = \gamma(T) - 1$. We show next that $\gamma_r(T') \leqslant \gamma_r(T) - 1$. If w is adjacent to a leaf z, then $z \in S$ and so, since S is a packing, $y \notin S$. On the other hand, if w is not a support vertex, then $S \cap N[w] = \{y\}$. In both cases, $S - \{u\}$ is a RDS of T'. Hence, $\gamma_r(T') \leqslant \gamma_r(T) - 1$. Thus, $\gamma_r(T') \geqslant \gamma(T') = \gamma(T) - 1 = \gamma_r(T) - 1 \geqslant \gamma_r(T')$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T') = \gamma_r(T')$ and $S - \{u\}$ is a $\gamma_r(T')$-set. Thus, T' is a (γ, γ_r)-tree. By the induction hypothesis, $T' \in \mathcal{T}$. By Lemma 3, $S_B(T')$ is the unique $\gamma_r(T')$-set. Thus, $S - \{u\} = S_B(T')$. Since $w \notin S$, the vertex w has status A in T'. Hence by operation $\mathcal{O}_1$, our labelling of T' can be extended to a labelling of T so that $T \in \mathcal{T}$. □

LEMMA 6. *If T is a (γ, γ_r)-tree, then T is a γ-excellent tree and $T \neq K_2$.*

Proof. By Theorem 8, $T \in \mathcal{T}$. By Theorem 3 in [11], the family $\mathcal{T}$ is a subfamily of the family of i-excellent trees, and so the tree $T \in \mathcal{T}$ is i-excellent. By Lemma 3, S_B is an independent dominating set of T, and so $|S_B| = \gamma(T) \leqslant i(T) \leqslant |S_B|$. Hence we must have equality throughout this inequality chain. In particular, $\gamma(T) = i(T)$. Thus the i-excellent tree T is also a γ-excellent tree. Since T is a (γ, γ_r)-tree, $T \neq K_2$. □

LEMMA 7. *If T is a γ-excellent tree and $T \neq K_2$, then T is a (γ, γ_r)-tree.*

Proof. We proceed by induction on the order n of a γ-excellent tree T. If $n = 1$, then the desired result holds. Since no star of order at least 3 is a γ-excellent tree, $\mathrm{diam}(T) \geqslant 3$. If $\mathrm{diam}(T) = 3$, then $T = P_4$ and the desired result holds. This establishes the base cases. Assume then that $n \geqslant 5$ and that every γ-excellent tree of order at least 3 and less than n is a (γ, γ_r)-tree. Let T be a γ-excellent tree of order n. Then, $\mathrm{diam}(T) \geqslant 4$. Let T be rooted at a leaf r of a longest path P. Let P be a r–u path, and let v be the neighbor of u. Further, let w denote the parent of v on this path, and let x denote the parent of w. Then, u is a leaf of T. Since T is γ-excellent, T has no strong support vertex. Hence, $\deg_T(v) = 2$.

We now consider three possibilities. In all cases, we prune the tree T to a γ-excellent tree T'. By the inductive hypothesis, T' is a (γ, γ_r)-tree. By Theorem 8, $T' \in \mathcal{T}$. By Lemma 3, there is a unique $\gamma_r(T')$-set that is a packing and contains all the leaves of T'. In each of the three cases, we let S' be such a $\gamma_r(T')$-set. We then show that S' can be extended to a RDS of cardinality $\gamma(T)$, whence T is a (γ, γ_r)-tree. Notice that should $T' = K_2$, then T is, in all cases, not a γ-excellent tree.

Case 1. Suppose $\mathbf{deg_T(w) = 2}$. Let $T' = T - \{u, v, w\}$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the vertex v, and so $\gamma(T) \leqslant \gamma(T') + 1$. On the other hand, let S be a $\gamma(T)$-set containing v. If $w \in S$, then we can simply replace the vertex w in S with the vertex x. Hence we may assume $w \notin S$. Thus, $S - \{v\}$ is a dominating set of T', and so $\gamma(T') \leqslant |S| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T') = \gamma(T) - 1$.

We show that T' is a γ-excellent tree. Let $z \in V(T')$. Since T is γ-excellent, there is a $\gamma(T)$-set S_z that contains the vertex z. If $u \in S_z$, then we can replace u in S_z with the vertex v. Hence we may assume $v \in S_z$. If $w \in S_z$, then we can replace the vertex w in S_z with the vertex x. Hence we may assume $w \notin S_z$. Thus, $S_z - \{v\}$ is a dominating set of T' that contains z. Since $|S_z| - 1 = \gamma(T) - 1 = \gamma(T')$, the vertex z is contained in a $\gamma(T')$-set. Since z is an arbitrary vertex of T', the tree T' is therefore a γ-excellent tree.

We show that $x \in S'$ (where S' is the $\gamma_r(T')$-set defined earlier). Let S_w be a $\gamma(T)$-set containing the vertex w. If $u \in S_w$, then we can replace u in S_w with the vertex v. Hence we may assume $v \in S_w$. By the minimality of the set S_w, $\mathrm{pn}(w, S_w) = \{x\}$. This implies that the vertex x is not a support vertex and has no child that is a support vertex. Hence every leaf-descendant of x is at distance 3 from x. Hence it follows from the properties of the set S' (that is both a RDS and a packing containing all leaves in T') that $x \in S'$ (irrespective of whether x is a leaf in T' or not).

Since $x \in S'$, the set $S' \cup \{u\}$ is a RDS of T, and so $\gamma(T) \leqslant \gamma_r(T) \leqslant |S'| + 1 = \gamma_r(T') + 1 = \gamma(T') + 1 = \gamma(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_r(T)$, and so T is a (γ, γ_r)-tree.

Case 2. Suppose $\deg_T(w) \geqslant 3$ and w is a support vertex. Let $T' = T - \{u, v\}$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the vertex v, and so $\gamma(T) \leqslant \gamma(T') + 1$. On the other hand, let S be a $\gamma(T)$-set containing w. We may assume $v \in S$. Thus, $S - \{v\}$ is a dominating set of T', and so $\gamma(T') \leqslant |S| - 1 = \gamma(T) - 1$. Consequently, $\gamma(T') = \gamma(T) - 1$.

We show that T' is a γ-excellent tree. Let $z \in V(T')$. Since T is γ-excellent, there is a $\gamma(T)$-set S_z that contains the vertex z. We may assume that $v \in S_z$. Since S_z contains either w or the leaf-neighbor of w, the set $S_z - \{v\}$ is a dominating set of T' that contains z. Since $|S_z| - 1 = \gamma(T) - 1 = \gamma(T')$, the vertex z is contained in a $\gamma(T')$-set, and so T' is a γ-excellent tree.

Since the $\gamma_r(T')$-set S' is a packing and contains all the leaves of T', the set S' contains the leaf-neighbor of w, and so $w \notin S'$. Hence the set $S' \cup \{u\}$ is a RDS of T, and so $\gamma(T) \leqslant \gamma_r(T) \leqslant |S'| + 1 = \gamma_r(T') + 1 = \gamma(T') + 1 = \gamma(T)$. Consequently, $\gamma(T) = \gamma_r(T)$ and T is a (γ, γ_r)-tree.

Case 3. Suppose $\deg_T(w) \geqslant 3$ and w is not a support vertex. Then each child of w is a support vertex of degree 2. Thus the maximal subtree T_w of T rooted at w is obtained from a star $K_{1,k}$, where $k = \deg_T(w) - 1 \geqslant 2$, by subdividing each edge exactly once.

Let $T' = T - V(T_w)$. Any $\gamma(T')$-set can be extended to a dominating set of T by adding to it the set $C(w)$ of k children of w, and so $\gamma(T) \leqslant \gamma(T') + k$. On the other hand, let S be a $\gamma(T)$-set containing the support vertices of T. Then, $C(w) \subset S$. If $w \in S$, then we can simply replace the vertex w in S with the vertex x. Hence we may assume $w \notin S$. Thus, $S - C(w)$ is a dominating set of T', and so $\gamma(T') \leqslant |S| - k = \gamma(T) - k$. Consequently, $\gamma(T') = \gamma(T) - k$.

We show that T' is a γ-excellent tree. Let $z \in V(T')$. Since T is γ-excellent, there is a $\gamma(T)$-set S_z that contains the vertex z. We may assume $C(w) \subset S_z$. If $w \in S_z$, then we can replace the vertex w in S_z with the vertex x. Hence we may assume $w \notin S_z$. Thus, $S_z - C(w)$ is a dominating set of T' that contains z. Since $|S_z - C(w)| = \gamma(T) - k = \gamma(T')$, the vertex z is contained in a $\gamma(T')$-set, and so T' is a γ-excellent tree.

We show that $x \in S'$. Let S_w be a $\gamma(T)$-set containing the vertex w. We may assume $C(w) \subset S_w$. By the minimality of the set S_w, $\mathrm{pn}(w, S_w) = \{x\}$. This implies that the vertex x is not a support vertex and has no child that is a support vertex. Hence every leaf-descendant of x is at distance 3 from x. Hence it follows from the properties of the set S' (that is both a RDS and a packing containing all leaves in T') that $x \in S'$ (irrespective of whether x is a leaf in T' or not).

Since $x \in S'$, the set S' can be extended to a RDS of T by adding to it all k leaf-descendants of w, and so $\gamma(T) \leqslant \gamma_r(T) \leqslant |S'| + k = \gamma_r(T') + k = \gamma(T') + k = \gamma(T)$. Consequently, we must have equality throughout this inequality chain. In particular, $\gamma(T) = \gamma_r(T)$, and so T is a (γ, γ_r)-tree. □

As an immediate consequence of Lemmas 3–7 we have our main result.

THEOREM 8. *Let T be a tree. Then the following statements are equivalent:*

(i) $T \in \mathcal{T}$;
(ii) *T has a unique $\rho(T)$-set and this set is a dominating set of T;*
(iii) *T is a (γ, γ_r)-tree;*
(iv) *T is γ-excellent and $T \neq K_2$.*

Proof. By Lemma 3, (i) ⇒ (ii). By Lemma 4, (ii) ⇒ (iii). By Lemma 5, (iii) ⇒ (i). By Lemma 6, (iii) ⇒ (iv). By Lemma 7, (iv) ⇒ (iii). □

We close with the remark that there do exist trees T with unique $\rho(T)$-sets that are not (γ, γ_r)-trees. For example, attach to each vertex of a path P_4 a pendant edge (the resulting tree is called the corona $\mathrm{coro}(P_4)$ of P_4) and then subdivide the edge joining the two vertices of maximum degree exactly once.

Acknowledgement

Research supported in part by the South African National Research Foundation and the University of KwaZulu-Natal.

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